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Polynomial super-$gl(n)$ algebras

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Abstract
We introduce a class of finite-dimensional nonlinear superalgebras $L = L_{\bar{0}} + L_{\bar{1}}$ providing gradings of $L_{\bar{0}} = gl(n) \simeq sl(n) + gl(1)$. Odd generators close by anticommutation on polynomials (of degree >1) in the $gl(n)$ generators. Specifically, we investigate 'type I' super-$gl(n)$ algebras, having odd generators transforming in a single irreducible representation of $gl(n)$ together with its contragredient. Admissible structure constants are discussed in terms of available $gl(n)$ couplings, and various special cases and candidate superalgebras are identified and exemplified via concrete oscillator constructions. For the case of the n-dimensional defining representation, with odd generators $Q_a, \bar{Q}^b$ and even generators $E^a{}_b, a, b = 1, \ldots, n$, a three-parameter family of quadratic super-$gl(n)$ algebras (deformations of $sl(n/1)$) is defined. In general, additional covariant Serre-type conditions are imposed in order that the Jacobi identities are fulfilled. For these quadratic super-$gl(n)$ algebras, the construction of Kac modules and conditions for atypicality are briefly considered. Applications in quantum field theory, including Hamiltonian lattice QCD and spacetime supersymmetry, are discussed.

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1. Introduction

The interplay between the application of symmetry principles to models of physical systems, and study of the classification, properties and representation theory of underlying algebraic structures, has long been a major theme in mathematical physics. A broad spectrum of generalized symmetry algebras is under active study, including infinite-dimensional algebras and superalgebras, deformations of universal enveloping algebras, and various ternary and other non-associative algebras. Although the general study of *nonlinear* Lie (super)algebras belongs to abstract deformation theory, in specific contexts enough structure exists to allow progress on classification and representation theory. For example, the so-called W-(super) algebras are rigidly constrained by their origins in Hamiltonian reduction of constrained systems on Lie–Poisson manifolds (for references see below).

In this spirit, we study in this paper a class of finite-dimensional nonlinear superalgebras, with attention to their covariant closure relations, and which are defined algebraically, without reference to additional structure. Namely, we consider superalgebras ($\mathbb{Z}_2$-graded algebras) $L = L_{\bar{0}} + L_{\bar{1}}$ with even subalgebra $L_{\bar{0}}$, and odd subalgebra $L_{\bar{1}}$, with defining relations of the form:

$$[L_{\bar{0}}, L_{\bar{0}}] \subseteq L_{\bar{0}} \qquad [L_{\bar{0}}, L_{\bar{1}}] \subseteq L_{\bar{1}} \qquad \{L_{\bar{1}}, L_{\bar{1}}\} \subseteq U(L_{\bar{0}}). \tag{1}$$

Such 'nonlinear super-$L_{\bar{0}}$ algebras' possess odd generators $L_{\bar{1}}$ whose anticommutation relations generalize the defining relations of Lie superalgebras in that they close only in the universal enveloping algebra $U(L_{\bar{0}})$, that is, on *polynomials* (of quadratic or higher degree) in the even generators $L_{\bar{0}}$. In this work we take the latter to be the classical Lie algebra $L_{\bar{0}} = gl(n) \simeq sl(n) + gl(1)$.

Study of the classification of such superalgebras devolves to examination of possible $L_{\bar{0}}$-modules $L_{\bar{1}}$, and admissible structure constants (1) which are consistent with the Jacobi identities. This is taken up in section 2 below, where we discuss the structure of candidate polynomial super-$gl(n)$ algebras of 'type I', that is, where $L_{\bar{1}}$ consists of the direct sum of an (arbitrary) irreducible $L_{\bar{0}}$-module $\{\lambda\}$ together with its contragredient representation $\{\bar{\lambda}\}$, denoted here by $gl_k(n/\{\lambda\} + \{\bar{\lambda}\})$ (where k is the maximal degree within $U(L_{\bar{0}})$ of the polynomials $[L_{\bar{1}}, L_{\bar{1}}]$). In general, many types of structure constant are, in principle, allowed (identified as tensor couplings). These must be enumerated in specific cases, and the Jacobi identities imposed in order to identify viable solutions. Examples include odd generators in totally antisymmetric and totally symmetric tensor representations, together with their contragredients. In section 3 concrete low-rank examples of this type are provided via explicit oscillator constructions, together with generalizations including parafermionic realizations.

The examples of section 3 fulfil the desired anticommutation relations because of the structure specific to the oscillator realizations. In section 4, a more complete treatment is given, for one special case, by examination of the quadratic super-$gl(n)$ algebras, with even generators $E^a{}_b$, $1 \leqslant a, b \leqslant n$ in the Gel'fand basis, and odd generators Q_a, $\bar{Q}^b$ in the *defining* n-dimensional representation of $gl(n)$, and its contragredient. A three-parameter family of quadratic algebras $gl_2(n/\{1\} + \{\bar{1}\})^{\mathsf{a},\alpha,\beta}$ is identified, which closely parallels the well-known *linear* super-$gl(n)$ algebra, namely the simple Lie superalgebra $sl(n/1) \equiv gl_1(n/\{1\} + \{\bar{1}\})$. In general, the Jacobi identities are satisfied provided additional covariant Serre-type relations of the form $E^a{}_b\bar{Q}^b = q\bar{Q}^a$, $Q_a E^a{}_b = Q_b q$ hold in the enveloping algebra, for some $gl(n)$-invariant $q = \alpha\langle E\rangle + \beta\mathbb{1}$, where $\langle E\rangle = E^c{}_c$. In section 4, an outline of the construction of Kac modules is also given, together with the derivation of a necessary condition for typicality.

In the concluding remarks (section 5), additional motivation for the investigation of polynomial superalgebras is discussed, in relation to symmetries of classical and quantum systems, including detailed comparisons with previous studies in the literature. (For reviews and relevant papers on the structure of the W-(super) algebras see [1–4]. Various contexts in which polynomial algebras have been introduced include classical and quantum mechanics and (quasi)-exactly solvable potentials [5–8], supersymmetric quantum mechanics [9], quantum many-body theory [10–12] and relativistic wave equations [13, 14]). Applications of the present work include supersymmetry between colour singlet baryon and meson states in Hamiltonian lattice QCD [15–19], and (for $n = 4$) new classes of conformal spacetime supersymmetries [20]. The appendix provides notational conventions for partition labelling of finite-dimensional irreducible tensor representations of $gl(n)$ (appendix A.1), and generalized

Gel'fand notation for the generators (appendix A.2). In appendix A.3 details of the fermionic oscillator construction for the case $gl_2(n/\{3\} + \{\bar{3}\})$ are given, allowing (indecomposable) modules to be identified in a $gl(n)$ basis, both on the fermionic Fock space, and via the adjoint action on the associated Clifford algebra (see tables 1 and 2 for the $n = 1$ and $n = 2$ cases). Finally (appendix A.4), for the case $n = 4$ the relation between the algebras $gl_2(4/\{1^3\} + \overline{\{1^3\}})$ (discussed in section 3) and the family $gl_2(4/\{1\} + \{\bar{1}\})^{\mathsf{a},\alpha,\beta}$ (section 4) is studied. (In addition to the above literature on physical applications of polynomial (super) algebras, the text of the paper contains extensive citations, for example, to provide background on group theory (see, for example, [21]; section 4, appendices A.1–A.3), salient references on parastatistics (section 3), and on the theory of characteristic identities for generators of simple Lie (super) algebras (sections 4 and 5, appendix A.2)).

2. Polynomial super-$gl(n)$ algebras $gl_k(n/\{\lambda\} + \{\bar{\lambda}\})$

In this section generic polynomial super-$gl(n)$ algebras will be studied from the point of view of admissible structure constants in the generalized sense. From the graded Jacobi identities (see section 4), the odd generators form an $L_{\bar{0}}$-module with respect to (the adjoint action of) the even subalgebra, and $\{L_{\bar{1}}, L_{\bar{1}}\}$ transforms under $\mathrm{ad}_{L_{\bar{0}}}$ in the (symmetric) tensor product $L_{\bar{1}} \otimes L_{\bar{1}}$ of the odd $L_{\bar{0}}$-module $L_{\bar{1}}$ with itself. Correspondingly, in view of the the Poincaré–Birkhoff–Witt theorem for the structure of the enveloping algebra, monomials in the even generators transform as symmetric tensor powers of the adjoint representation. Thus generalized structure constants[3] can only exist with the correct polynomial degree k if the corresponding *symmetric* kth tensor power of the adjoint representation $\mathrm{ad}_{L_{\bar{0}}}$ contains common irreducible submodules, with the branching multiplicity of the latter determining their number and type.

Study of the classification of such superalgebras devolves to examination of possible $L_{\bar{0}}$-modules $L_{\bar{1}}$, and admissible structure constants (1). Similar questions arise in the study of simple Lie superalgebras [22] (where $L_{\bar{0}}$ replaces $U(L_{\bar{0}})$ in (1) above). An analogous situation is addressed in the Witt construction [23], where one considers Lie algebras associated with a given Lie algebra $L_{\bar{0}}$ extended by a certain $L_{\bar{0}}$-module (a trivial extension being the semidirect product, with the module given the structure of an Abelian algebra)[4]. These considerations are more tractable if we turn to the 'type I' super-$gl(n)$ algebras: the $L_{\bar{0}}$-module $L_{\bar{1}}$ is the sum of a single irreducible representation and its contragredient. The $\mathbb{Z}_2$-grading is thus inherited from a $\mathbb{Z}$-grading of L, associated with the spectrum of the adjoint action of the abelian summand of $gl(n) \simeq sl(n) + gl(1)$. Thus $L = L_{-1} + L_0 + L_1$, with $L_{\bar{0}} = L_0$ and $L_{\bar{1}} = L_{-1} + L_1$. This entails $\{L_{\pm 1}, L_{\pm 1}\} = 0$, and $\{L_{+1}, L_{-1}\} \subset U(L_0)$. Without loss of generality, we may assume that L_{+1} is an irreducible representation of the semisimple part $sl(n)$, with L_{-1} the corresponding contragredient. Now for semisimple Lie algebras, Joseph's theorem [25] states that the enveloping algebra $U(L_0)$ is isomorphic as an L_0-module, to the sum over all dominant integral weights, of the tensor product of the corresponding (finite-dimensional) highest weight module, with its contragredient. Thus, it is possible to investigate whether the anticommutators $\{L_{+1}, L_{-1}\}$ can be associated with a unique element of $U(L_0)$.

[3] If $\{T_a\}$ is a basis for the even subalgebra $L_{\bar{0}}$ and $\{Q_\alpha\}$ a basis for $L_{\bar{1}}$, then the (anti)commutation relations (1) take the form

$$[T_a, T_b] = f_{ab}{}^c T_c \qquad [T_a, Q_\beta] = f_{a\beta}{}^\gamma Q_\gamma \qquad \{Q_\alpha, Q_\beta\} = f_{\alpha\beta}{}^{c_1 c_2 \cdots} T_{c_1} T_{c_2} \ldots + \cdots$$

where there may be lower degree terms in the last line. Thus $\{Q_\alpha\}$ form a tensor operator under the action of $L_{\bar{0}}$ and the anticommutator $\{Q_\alpha, Q_\beta\}$ transforms in the tensor product of the relevant representations of the even subalgebra.

[4] This construction has recently been considered in connection with embeddings of quantum algebras [24].

However, as Joseph's theorem does not mandate any relation between the polynomial degree within $U(L_0)$ of elements of a given tensor product contributing to the sum, we proceed more generally. Let, then, $\{\lambda\}$ denote a dominant integral weight of $gl(n)$ and $\{\bar\lambda\}$ the corresponding contragredient; where no confusion arises, these symbols will also stand for the character of the corresponding irreducible, finite-dimensional highest weight module (for notation see appendix A.1)[5]. The adjoint representation of $gl(n)$ is the reducible representation $\{\bar 1\}\cdot\{1\}$ (corresponding to the tensor product of the n-dimensional defining representation with its contragredient) with irreducible parts $\{\bar 1; 1\} + \{0\}$ reflecting the reduction to $sl(n) + gl(1)$. According to the previous discussion, distinct types of structure constant will be determined by the branching multiplicity of those irreducible components of the symmetric kth tensor power of the adjoint representation, which are in common with the irreducible modules occurring in the decomposition of the tensor product $\{\bar\lambda\}\cdot\{\lambda\}$. Let ℓ be the weight of $\{\lambda\}$ as a partition and k be the polynomial degree of nonlinearity in the enveloping algebra of $gl(n)$ characterizing the algebra. Then we have

$$\{\bar\lambda\}\cdot\{\lambda\} = \sum_{\mu,\nu} n^{\lambda}{}_{\mu\nu}\{\bar\mu;\nu\} \qquad (\{\bar 1\}\cdot\{1\})\otimes\{k\} = \sum_{\mu,\nu} n^{k}{}_{\mu\nu}\{\bar\mu;\nu\}. \tag{2}$$

The multiplicities $n^k{}_{\mu\nu}, n^\lambda{}_{\mu\nu}$ are defined in appendix A.1 in terms of the standard Littlewood–Richardson coefficients. In appendix A.1 it is shown that $n^k{}_{\mu\nu} \geqslant n^\lambda{}_{\mu\nu}$ provided $k \geqslant \ell$, and moreover if $\ell > k$, then $n^k{}_{\mu\nu} = 0$ for $\mu,\nu \vdash k+1,\ldots,\ell$. In the latter case, generalized structure constants for the corresponding symmetry types $\{\bar\mu;\nu\}$ arising from $\{\bar\lambda\}\cdot\{\lambda\}$ do not exist.

To complete the discussion of couplings in specific cases, it is necessary to adopt an explicit notation. Recall the well-known presentation of $gl(n)$ via the Gel'fand generators $E^a{}_b$, $1 \leqslant a, b \leqslant n$ (see, for example, the first line of (10) below). Define the matrix powers $(E^{k+1})^a{}_b = (E^k)^a{}_c E^c{}_b$ in the obvious way, and their traces $\langle E^k\rangle = (E^k)^c{}_c$ (the standard Casimir invariants, see appendix A.2). Bearing in mind Joseph's theorem [25], the following generalized permanents [28] provide a suitable spanning set for $U(gl(n))$: for $\{\lambda\} \vdash \ell$, and $\{\mu\}$ an ℓ-part partition (of weight k, with nonzero parts augmented by zeroes as necessary), define

$$[\{\lambda\};\{\mu\}]^{a_1a_2\ldots a_\ell}{}_{b_1b_2\ldots b_\ell} = \frac{1}{\ell!^2}\sum_{\rho,\sigma\in S_\ell}\chi^\lambda(\rho\sigma^{-1})(E^{\mu_1})^{a_{\sigma1}}{}_{b_{\rho1}}(E^{\mu_2})^{a_{\sigma2}}{}_{b_{\rho2}}\ldots(E^{\mu_\ell})^{a_{\sigma\ell}}{}_{b_{\rho\ell}}. \tag{3}$$

Here χ^λ is the irreducible character of S_ℓ corresponding to the class λ, and $(E^0)^a{}_b \equiv \delta^a{}_b$. Thus $\{\lambda\}$ determines the symmetry type $\{\bar\lambda\}\cdot\{\lambda\}$, and $\{\mu\}$ the distribution of tensor contractions, for polynomials in the generators $E^a{}_b$ belonging to the enveloping algebra $U(gl(n))$. In this notation the matrix powers are, of course, $(E^k)^a{}_b \equiv [\{1\};\{k\}]^a{}_b$, while the Casimir operators are simply found by contraction of any $[\{\lambda\};\{\mu\}]$ with

$$\Delta_\lambda{}^{a_1a_2\ldots a_\ell}{}_{b_1b_2\ldots b_\ell} = [\{\lambda\};\{0\}]^{a_1a_2\ldots a_\ell}{}_{b_1b_2\ldots b_\ell}.$$

For the cases considered in the following, elementary tensor notation suffices for explicit constructions. As an example let us analyse in detail the case $k = \ell = 3$ and $\{\lambda\} = \{2, 1\}$.

[5] We label highest weight representations, and where no confusion arises their corresponding characters, by partitions $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ corresponding to the symmetry type of irreducible tensors. Partition labelling for irreducible representations of simple Lie algebras is developed in [26]. Aspects of this notation for $gl(n)$ and $sl(n)$, including the various 'products' $\cdot$, $\otimes$, $\circ$, are discussed in appendix A.1. In particular, a composite partition $\{\bar\rho;\sigma\}$ corresponds to an irreducible tensor of contravariant symmetry type ρ, and covariant symmetry type σ, and is traceless with respect to contractions between contravariant and covariant indices. (See appendix A.1, and also [27].)

Explicitly, we have (see appendix A.1)

$$
\begin{aligned}
&\{\overline{2,1}\}\cdot\{2,1\} = \{\overline{2,1}; 2,1\} + \{\bar{2}; 2\} + 2\{\bar{1}; 1\} + \{0\} \\
&(\{\bar{1}\}\cdot\{1\})\otimes\{3\} = \{\bar{3}; 3\} + \{\overline{2,1}; 2,1\} + \{\overline{1,1,1}; 1,1,1\} \\
&\qquad + 2\{\bar{2}; 2\} + 2\{\overline{1,1}; 1,1\} + 4\{\bar{1}; 1\} + 3\{0\}
\end{aligned}
\tag{4}
$$

from which, at the level of reduced matrix elements, at degree three, the putative cubic algebra $gl_3(n/\{2,1\} + \{\overline{2,1}\})$ has 10 types of structure constant (or nine free parameters for the associated reduced matrix elements, up to overall normalization), corresponding to the maximum multiplicities of the common irreducible components of the above two decompositions (excluding additional structure constants arising from lower degree). To complete the construction of couplings for this case, define the following objects in the enveloping algebra:

$$
\{F\cdot F'\cdot F''\}^{abc}{}_{pqr} \equiv \left(F^a{}_p F'^b{}_q + F^b{}_p F'^a{}_q\right)F''^c{}_r - \left(F^c{}_p F'^b{}_q + F^a{}_p F'^c{}_q\right)F''^a{}_r + \cdots \tag{5}
$$

being of *mixed* symmetry type $[\{2,1\}; \{\mu_1, \mu_2, \mu_3\}]$ with respect to contravariant and covariant indices (where '$\cdots$' represents three additional quartets of terms establishing mixed symmetry with respect to the pqr label permutations (the terms shown explicitly possess mixed symmetry with respect to abc). F, F', F'' stand for $E^{\mu_1}, E^{\mu_2}, E^{\mu_3}$, respectively with $\mu_1+\mu_2+\mu_3 = 3$). Such terms exist by (2), and the 10 couplings at cubic degree required by (4) are schematically[6]

$$
\begin{aligned}
&\{E\cdot E\cdot E\} \qquad \{E^2\cdot E\cdot\delta\} \qquad \{E\cdot E\cdot\delta\}\langle E\rangle \\
&\{E^3\cdot\delta\cdot\delta\} \qquad \{E^2\cdot\delta\cdot\delta\}\langle E\rangle \qquad \{E\cdot\delta\cdot\delta\}\langle E^2\rangle \qquad \{E\cdot\delta\cdot\delta\}\langle E\rangle^2 \\
&\{\delta\cdot\delta\cdot\delta\}\langle E^3\rangle \qquad \{\delta\cdot\delta\cdot\delta\}\langle E^2\rangle\langle E\rangle \qquad \{\delta\cdot\delta\cdot\delta\}\langle E\rangle^3
\end{aligned}
\tag{6}
$$

where $\{\cdots\}$ indicates mixed permutation symmetry as in (5) above.

It is noteworthy that the 'leading' irreducible component $\{\overline{2,1}; 2,1\}$ has unit multiplicity in both expansions in (4) above, corresponding to a single reduced matrix element (which can be set to 1 by a choice of overall normalization); from appendix A.1, it is apparent that $n^{\lambda}{}_{\lambda\lambda} = 1$ in general (so the same count of the 'leading' component holds whenever $\ell = k$). This circumstance is intimately related to a canonical construction (outlined in appendix A.2, valid for all simple Lie algebras), in which the generators and defining relations of $gl(n)$ may be presented in terms of components $\mathcal{E}^{abc\ldots}{}_{pqr\ldots}$ of an arbitrary tensor operator (of any rank and symmetry type), and the defining relations presented in a manner consistent with this.

3. Low-rank examples and oscillator constructions

The general discussion of the previous section has identified a large class of candidate nonlinear super-$gl(n)$ algebras on the basis of structure constants which are admissible on the grounds of $gl(n)$ invariance. This guarantees the validity of the Jacobi identity involving $[L_{\bar{0}}, \{L_{\bar{1}}, L_{\bar{1}}\}]$. The remaining odd Jacobi identity involving $[L_{\bar{1}}, \{L_{\bar{1}}, L_{\bar{1}}\}]$ can be best addressed in specific low-dimensional cases, or via explicit constructions, to which we now turn.

Consider, for example, the simplest case $n = 1$, and the polynomial superalgebra $gl_k(1/\{\ell\} + \{\bar{\ell}\})$. The Abelian algebra $gl(1)$ has a single generator K, and one-dimensional representations (labelled as type $\{\ell\}$).[7] With odd generators denoted $\bar{Q}$ and Q we have the defining relations (taking $\ell = 1$ without loss of generality)

$$
[K, Q] = -Q \qquad [K, \bar{Q}] = \bar{Q} \qquad \{\bar{Q}, Q\} = f(K)
$$

[6] Traceless forms of the objects (5) can, of course, be constructed if required.

[7] Here ℓ is simply an additive charge quantum number.

for some polynomial f of degree k, together with $\{Q, Q\} = \{\bar{Q}, \bar{Q}\} = 0$. The graded Jacobi identity entails

$$[Q, \{\bar{Q}, Q\}] = [\{Q, \bar{Q}\}, Q] - [\bar{Q}, \{Q, Q\}] = [\{Q, \bar{Q}\}, Q]$$

whereupon $f(K)$ is central, $f(K) \equiv H$. Thus, we have trivially regained the structure of the Lie superalgebra $gl(1/1)$ (the algebra of supersymmetric quantum mechanics), with defining relations

$$\begin{aligned} &[K, Q] = -Q && [K, \bar{Q}] = +\bar{Q} \\ &[H, Q] = 0 && [H, \bar{Q}] = 0 \\ &[H, K] = 0 \qquad \text{and} && \{\bar{Q}, Q\} = H. \end{aligned}$$

For the remainder of this section we consider examples of polynomial super-$gl(n)$ algebras via concrete oscillator realizations, in which there is enough structure to evaluate the anticommutator of odd generators explicitly. This will verify, for these cases, the general analysis of section 2 above, and at the same time guarantee all Jacobi identities. Thus, we take generating sets a_i, a^i, $i = 1, 2, \ldots, n$, b_j, b^j, $j = 1, 2, \ldots, n$ of either fermionic or bosonic creation and annihilation operators, respectively, where $a^i \equiv {a_i}^\dagger$, $b^i \equiv {b_i}^\dagger$, satisfying the canonical (anti-)commutation relations

$$\{a_i, a^j\} = \delta_i{}^j \mathbb{1} \qquad \{a_i, a_j\} = \{a^i, a^j\} = 0 \tag{7}$$

$$[b_i, b^j] = \delta_i{}^j \mathbb{1} \qquad [b_i, b_j] = [b^i, b^j] = 0 \tag{8}$$

respectively (below we also consider parastatistics realizations). The even generators for $gl(n)$ are the usual quadratic combinations giving the Gel'fand basis,

$$E^i{}_j = a^i a_j + \text{const}\,\mathbb{1} \qquad \text{or} \qquad E^i{}_j = b^i b_j + \text{const}\,\mathbb{1} \tag{9}$$

with

$$\left[E^i{}_j, E^k{}_\ell\right] = \delta_j{}^k E^i{}_\ell - \delta^i{}_\ell E^k{}_j \qquad \left[E^i{}_j, c^k\right] = \delta_j{}^k c^i \qquad \left[E^i{}_j, c_k\right] = -\delta^i{}_k c_j \tag{10}$$

where $c \equiv a$ *or* b. The odd generators of the nonlinear superalgebras will be composites in the bosonic and fermionic oscillator modes, transforming in tensor representations of various types. In particular, monomials purely in fermionic or bosonic creation operators are automatically antisymmetric or symmetric in permutations of their mode labels due to their mutual anticommutativity or commutativity, respectively (monomials in the corresponding annihilation operators transform contragrediently). It is thus natural to consider such monomials as candidates for odd generators of super-$gl(n)$ algebras.

To set the context for the generalizations under investigation, we consider firstly the rank 1 and 2 cases, with either $\{\lambda\} = \{1^\ell\}$ (fermions), or $\{\lambda\} = \{\ell\}$ (bosons), $\ell = 1, 2$. Surprisingly perhaps, at this algebraic level, particle statistics does not preclude utilization of bosonic oscillators as odd generators, and it will turn out that the $k = 1$ cases are well-known constructions (there are also $k = 2$ generalizations). To make the discussion complete, we also consider polynomial algebras with generators closing on commutation relations, and also polynomial superalgebras with even part *larger* than $gl(n)$.

The rank $\ell = 1$ case is familiar in the linear situation, and corresponds simply to enlarging the bilinear oscillator $gl(n)$ generators by appending the mode operators themselves. Take firstly the bosonic generators under the 'natural' commutator bracket relations. Clearly $b^i, b_j, \mathbb{1}$ and $E^i{}_j$ (see (8), (10) and (9) above) generate the standard semidirect product of the Weyl–Heisenberg algebra with the automorphism algebra $gl(n)$ (extendible to $sp(2n)$, see below). In the context of *nonlinear* constructs, there is also the remarkable Holstein–Primakoff–Dyson [29] realization, which 'dresses' the oscillator modes so that linear closure on $gl(n+1)$ is obtained:

$$E^{n+1}{}_i = \sqrt{p\mathbb{1} + \langle E\rangle}\, b_i \qquad E^i{}_{n+1} = b^i\sqrt{p\mathbb{1} + \langle E\rangle}$$

for some parameter p.

As is well known, the same bosonic modes are natural candidates for odd generators of a superalgebra. In this case, a super-$gl(n)$ algebra is not achieved, as the 'unnatural' choice of anticommutator will generate new operators $\bar{S}^{ij} \equiv \frac{1}{2}\{b^i, b^j\}$, $S_{ij} \equiv \frac{1}{2}\{b_i, b_j\}$, as well as $E^i{}_j \equiv \frac{1}{2}\{b^i, b_j\}$ (see (9) above). However, instead closure is achieved on the enlarged automorphism algebra $sp(2n)$ in the 'even' (bilinear) generators, and hence on the natural superalgebra $osp(1/2n)$ including the oscillator modes themselves.

The situation with fermionic oscillators is entirely parallel to the bosonic case. Choosing closure of the oscillator modes by anticommutators reproduces the canonical anticommutation relations, giving the complex Dirac or Clifford algebra generated by $a^i, a_j, \mathbb{1}$ together with automorphisms generated by $E^i{}_j$ (extendible to $so(2n)$, see below). Again, there is a construction analogous to the Holstein–Primakoff–Dyson realization [30–33], this time formally polynomial rather than in an extension of the enveloping algebra, whereby the fermionic generators can be appropriately 'dressed' so as to achieve closure, this time on the Lie superalgebra $sl(n/1)$:

$$E^{n+1}{}_i = \sqrt{p\mathbb{1} - \langle E\rangle}\, a_i \qquad E^i{}_{n+1} = a^i\sqrt{p\mathbb{1} - \langle E\rangle}$$

for some parameter p. Finally the choice of 'unnatural' *commutator* brackets for the fermionic modes will generate, as well as $E^i{}_j \equiv \frac{1}{2}[a^i, a_j]$ (see (9) above), new operators $\bar{A}^{ij} \equiv \frac{1}{2}[a^i, a^j]$, $A_{ij} \equiv \frac{1}{2}[a_i, a_j]$ which together with a^i, a_j close on the enlarged automorphism algebra $so(2n+1)$. For the choice of *commutation* relations, the rank $\ell = 2$ case is subsumed in the above discussion of $\ell = 1$, in that closure on $sp(2n) \supset gl(n), so(2n) \supset gl(n)$ was already found for bosons and fermions respectively via the symmetric and antisymmetric rank 2 tensors S and A.

With the exception of the Holstein–Primakoff–Dyson realization and its superalgebra analogue, all examples so far have been for up to quadratic realizations of classical Lie (super) algebras ($k = 1$). This subject can be refined to deal with many cases of subalgebra chains, and especially to discuss real forms [34, 35]. On the other hand, the behaviour of the S and A tensors under anticommutation with their contragredients furnishes a first example of polynomial superalgebras, although not of $\mathbb{Z}_2$ graded super-$gl(n)$ type:

$$\begin{aligned} \{\bar{S}^{(ij)}, S_{(pq)}\} &= \tfrac{1}{2}(E\cdot E)^{(ij)}{}_{(pq)} + \tfrac{1}{2}(E\cdot\delta)^{(ij)}{}_{(pq)} + \delta^{(ij)}{}_{(pq)} \\ \{\bar{A}^{[ij]}, A_{[pq]}\} &= -\tfrac{1}{2}[E\cdot E]^{[ij]}{}_{[pq]} + \tfrac{1}{2}[E\cdot\delta]^{[ij]}{}_{[pq]} - \delta^{[ij]}{}_{[pq]} \end{aligned} \tag{11}$$

where $(E\cdot E)$, $(E\cdot\delta)$, $[E\cdot E]$ and $[E\cdot\delta]$ are the strength 1 minimal combinations[8] possessing the appropriate (anti)symmetry (compare (3) above),

$$\begin{aligned} (E\cdot E)^{(ij)}{}_{(pq)} &= \left(E^i{}_p E^j{}_q + E^j{}_p E^i{}_q + E^i{}_q E^j{}_p + E^j{}_q E^i{}_p\right) \\ (E\cdot\delta)^{(ij)}{}_{(pq)} &= \left(E^i{}_p \delta^j{}_q + E^j{}_p \delta^i{}_q + E^i{}_q \delta^j{}_p + E^j{}_q \delta^i{}_p\right) \\ \delta^{(ij)}{}_{(pq)} &= \delta^i{}_p\delta^j{}_q + \delta^i{}_q\delta^j{}_p \\ [E\cdot E]^{[ij]}{}_{[pq]} &= \left(E^i{}_p E^j{}_q - E^j{}_p E^i{}_q - E^i{}_q E^j{}_p + E^j{}_q E^i{}_p\right) \\ [E\cdot\delta]^{[ij]}{}_{[pq]} &= \left(E^i{}_p \delta^j{}_q - E^j{}_p \delta^i{}_q - E^i{}_q \delta^j{}_p + E^j{}_q \delta^i{}_p\right) \\ \delta^{[ij]}{}_{[pq]} &= \delta^i{}_p\delta^j{}_q - \delta^i{}_q\delta^j{}_p. \end{aligned}$$

[8] For simplicity, the $gl(n)$ generators are defined as $E^i{}_j \equiv b^i b_j$, $E^i{}_j \equiv a^i a_j$ in these equations (see (9)).

Including also the mutual *commutativity* of the tensor components of each of these tensors separately, gives a structure of mixed grading resembling a nonlinear colour (super) algebra[9].

For the remaining two concrete examples in this section, we move to the rank $\ell = 3$ case, with fermionic oscillator realizations for either $\{\lambda\} = \{1^3\}$, $k = 2$ or $\{\lambda\} = \{3\}$, $k = 2$. These two cases typify several infinite families of super-$gl(n)$ algebras with analogous structure (in which k grows with ℓ). Further examples, including $\{\lambda\} = \{2, 1\}$ at rank 3 in a parafermionic construction, and higher-dimensional bosonic cases, are discussed in more general terms in the conclusion of this section.

For $gl_2(n/\{1^3\} + \overline{\{1^3\}})$ define the odd generators

$$\bar{Q}^{[ijk]} = a^i a^j a^k \qquad Q_{[pqr]} = a_p a_q a_r \tag{12}$$

together with

$$E^i{}_j \equiv a^i a_j \tag{13}$$

(see (9)). Then, in appropriately symmetrized tensor notation, the defining relations of this realization of $gl_2(n/\{1^3\} + \overline{\{1^3\}})$ become[10]

$$\begin{aligned}
&\{\bar{Q}^{[ijk]}, Q_{[pqr]}\} = -\tfrac{1}{4}[E\cdot E\cdot\delta]^{[ijk]}{}_{[pqr]} + \tfrac{1}{2}[E\cdot\delta\cdot\delta]^{[ijk]}{}_{[pqr]} - \delta^{[ijk]}{}_{[pqr]} \\
&\{Q^{[ijk]}, Q^{[pqr]}\} = 0 \qquad \{Q_{[ijk]}, Q_{[pqr]}\} = 0 \\
&\left[E^i{}_j, \bar{Q}^{[k\ell m]}\right] = \delta_j{}^k \bar{Q}^{[i\ell m]} + \delta_j{}^\ell \bar{Q}^{[kim]} + \delta_j{}^m \bar{Q}^{[k\ell i]} \\
&\left[E^i{}_j, Q_{[pqr]}\right] = -\delta^i{}_p Q_{[jqr]} - \delta^i{}_q Q_{[pjr]} - \delta^i{}_r Q_{[pqj]}.
\end{aligned} \tag{14}$$

Here $[E\cdot E\cdot\delta]$ and $[E\cdot\delta\cdot\delta]$ are the appropriate strength 1 minimal combinations possessing the required antisymmetry (compare (3) above),

$$\begin{aligned}
&[E\cdot E\cdot\delta]^{[ijk]}{}_{[pqr]} = \left(E^i{}_p E^j{}_q - E^j{}_p E^i{}_q - E^i{}_q E^j{}_p + E^j{}_q E^i{}_p\right)\delta^k{}_r + \cdots \\
&[E\cdot\delta\cdot\delta]^{[ijk]}{}_{[pqr]} = E^i{}_p\left(\delta^j{}_q\delta^k{}_r - \delta^j{}_r\delta^k{}_q\right) + \cdots \\
&\delta^{[ijk]}{}_{[pqr]} = \delta^i{}_p\left(\delta^j{}_q\delta^k{}_r - \delta^j{}_r\delta^k{}_q\right) + \cdots.
\end{aligned} \tag{15}$$

Allowing for cyclic permutations on ijk and pqr, $[E\cdot E\cdot\delta]$ contains a total of $4\times 9 = 36$ terms, $[E\cdot\delta\cdot\delta]$ contains $9\times 2 = 18$ terms and $[\delta\cdot\delta\cdot\delta]$ just $3\times 2 = 6$ terms.

For $gl_2(n/\{3\} + \{\bar{3}\})$ introduce $m = 3n$ and corresponding fermionic oscillators a^{iA}, a_{jB}, $i, j = 1, \ldots, n$ and $A, B = 1, 2, 3$. We take (see (9))

$$E^{iA}{}_{jB} \equiv a^{iA} a_{jB} \qquad E^i{}_j = E^{iA}{}_{jA} \qquad F^A{}_B = E^{iA}{}_{iB} \tag{16}$$

for the generators of $gl(3n)$, $gl(n)$ and colour $gl(3)$, respectively. Then with the help of the totally antisymmetric alternating tensor ϵ_{ABC} define

$$\bar{W}^{(ijk)} = \epsilon_{ABC} a^{iA} a^{jB} a^{kC} \qquad W_{(pqr)} = \epsilon^{ABC} a_{pA} a_{qB} a_{rC}. \tag{17}$$

[9] The natural graded structure when both fermionic and bosonic oscillator modes are present, where closure on *both* linear and bilinear combinations is required, is indeed that of a $\mathbb{Z}_2 \times \mathbb{Z}_2$ graded colour superalgebra. (See [36] and references therein.)

[10] Following the arguments of section 2 above, in this case there are potentially eight couplings, or seven arbitrary coefficients up to normalization (see appendix A.1 and (20)).

Clearly, the 'colour singlet' combinations $\bar{W}^{(ijk)}$ and $W_{(pqr)}$ are *symmetric* in the mode labels, and elementary calculation leads to the following, appropriately symmetrized, $gl_2(n/\{3\}+\{\bar{3}\})$ defining relations (see footnote 10) :

$$
\begin{aligned}
&\{W^{(ijk)}, W_{(pqr)}\} = \tfrac{1}{2}(E\cdot E\cdot\delta)^{(ijk)}{}_{(pqr)} + 2(E\cdot\delta\cdot\delta)^{(ijk)}{}_{(pqr)} - 6\delta^{(ijk)}{}_{(pqr)} \\
&\{W^{(ijk)}, W^{(pqr)}\} = 0 \qquad \{W_{(ijk)}, W_{(pqr)}\} = 0 \\
&\left[E^i{}_j, \bar{W}^{(k\ell m)}\right] = \delta_j{}^k \bar{W}^{(i\ell m)} + \delta_j{}^\ell \bar{W}^{(kim)} + \delta_j{}^m \bar{W}^{(k\ell i)} \\
&\left[E^i{}_j, W_{(pqr)}\right] = -\delta^i{}_p W_{(jqr)} - \delta^i{}_q W_{(pjr)} - \delta^i{}_r W_{(pqj)}.
\end{aligned}
\tag{18}
$$

Here $(E\cdot E\cdot\delta)$ and $(E\cdot\delta\cdot\delta)$ are the appropriate strength 1 minimal combinations possessing the required *symmetry* type (compare (3), (15) above),

$$
\begin{aligned}
&(E\cdot E\cdot\delta)^{(ijk)}{}_{(pqr)} = \left(E^i{}_p E^j{}_q + E^j{}_p E^i{}_q + E^i{}_q E^j{}_p + E^j{}_q E^i{}_p\right)\delta^k{}_r + \cdots \\
&(E\cdot\delta\cdot\delta)^{(ijk)}{}_{(pqr)} = E^i{}_p\left(\delta^j{}_q\delta^k{}_r + \delta^j{}_r\delta^k{}_q\right) + \cdots \\
&\delta^{(ijk)}{}_{(pqr)} = \delta^i{}_p\left(\delta^j{}_q\delta^k{}_r + \delta^j{}_r\delta^k{}_q\right) + \cdots.
\end{aligned}
\tag{19}
$$

Allowing for cyclic permutations on ijk and pqr, $(E\cdot E\cdot\delta)$ contains a total of $4\times 9 = 36$ terms, $(E\cdot\delta\cdot\delta)$ contains $9\times 2 = 18$ terms and $(\delta\cdot\delta\cdot\delta)$ just $3\times 2 = 6$ terms.

We close this section with some general remarks on ways of furnishing further specific constructions of polynomial superalgebras. It is clear that the two rank 3 fermionic examples generalize to arbitrary rank. In the antisymmetric $\{\lambda\} = \{1^\ell\}$ case, choose degree ℓ monomials in the fermionic creation and annihilation operators (giving anticommutators closing at degree $k = \ell - 1$ in $E^i{}_j$), and in the symmetric $\{\lambda\} = \{\ell\}$ case, choose $m = \ell n$ and define (for odd degree ℓ) colour singlet monomials with the appropriate rank ℓ alternating tensor, with anticommutators closing on degree $k = \ell - 1$ in $E^i{}_j$.

More generally, contractions with *any* suitable invariant tensor yield a plethora of possible symmetry types $\{\lambda\}$ for putative odd generators. For example, if $m = pn$ and a symmetric bilinear form (metric tensor) of dimension p exists, it is known that tensor powers yield symmetry types corresponding to all partitions $\{\lambda'\}$ of even row lengths (see appendix A.1). The corresponding tensor contraction against a totally antisymmetric monomial

$$a^{i_1A_1}\ldots a^{i_\ell A_\ell}$$

thus yields a fermionic tensor of $gl(n)$ symmetry type $\{\lambda\}$ (with even column lengths) corresponding to the transpose of the partition $\{\lambda'\}$ (see appendix A.1). Similar remarks apply to tensors constructed from an available *antisymmetric* bilinear form (with p even). However, as ℓ is necessarily even, closure with anticommutators is 'unnatural', and the candidate polynomial superalgebra is of $so(2n)$ type in this case. In principle, similar purely bosonic constructions are available. From the discussion of the rank 1 and 2 cases, at rank 3 in the symmetric case, a polynomial super-$sp(2n)$ algebra of the type $sp_3(2n/\langle 3\rangle)$ can be expected, with corresponding higher-dimensional generalizations. A more general approach, but beyond the scope of the present work, is to appeal to the well-developed theory of general tensor invariants of arbitrary rank and symmetry type [37–40].

Despite the above flexibility and ubiquity in oscillator constructions, it is nonetheless difficult to see how a natural candidate polynomial superalgebra for *odd* rank *mixed* symmetry tensor types is possible (few algebras have natural primitive invariants of this type). A final possibility worth pointing out is the use of parafermi or parabose oscillator realizations [41, 42].[11] For example, for parafermions of order p, monomials of permutation symmetry

[11] Modular statistics also provide representations of colour algebras and superalgebras of general permutation symmetry type [43].

type of up to p columns in the mode labels can be constructed; indeed the fundamental trilinear relation

$$[a^i, [a^j, a^k]] = 0$$

(for any order of parafermi statistics) simply ensures that the combination $a^i[a^j, a^k]$ is automatically of mixed [44] symmetry type $\{2, 1\}$. Furthermore, the structure of the parafermionic oscillator enveloping algebra is such [45] that even monomials can be represented as polynomials in the bilinears $[a^i, a^j]$, $[a^k, a_\ell]$, $[a_m, a_n]$, which are known to generate $so(2n)$ just as in the fermionic case [45]. Thus, the mixed symmetry rank 3 case realized by parafermions may be a candidate $so_3(2n/[2, 1])$ polynomial superalgebra[12]. For parafermi statistics of order p, the corresponding generalized Fock space realization [45] would form a submodule of the spinor representation $\left[\frac{1}{2}p, \frac{1}{2}p, \ldots, \pm\frac{1}{2}p\right]$ of $so(2n)$.

4. Quadratic super-$gl(n)$ algebras $gl_2(n/\{1\} + \{\bar{1}\})$

In this section we develop a more complete treatment of a single class of polynomial super-$gl(n)$ algebras than has been possible for the more general cases. We return to analogues of the simple Lie superalgebra $sl(n/1)$, wherein the even part $gl(n) \simeq sl(n) + gl(1)$ is graded by odd generators in the irreducible n-dimensional *defining* representation and its contragredient. In the present notation, we have $sl(n/1) \equiv gl_1(n/\{1\} + \{\bar{1}\})$. As shown in section 3 above, the linear case is familiar from elementary oscillator constructions, but we concentrate here on the quadratic generalization, $gl_2(n/\{1\} + \{\bar{1}\})$, which cannot be so realized (except for the special case $n = 4$, see below). Below, a complete account is given of structure constants and defining relations, followed by a discussion of certain classes of irreducible representations of these quadratic superalgebras.

Following the previous discussion of classes of structure constants, in order to write the nonvanishing anticommutator of the odd generators in the most general way, allowing for terms of degree 0, 1 and 2 in the $gl(n)$ enveloping algebra, the following decompositions should be noted (see (4) and section 2):

$$\begin{aligned}
&\{\bar{1}\} \cdot \{1\} = \{\bar{1}; 1\} + \{0\} \\
&\{\bar{1}; 1\} \otimes \{2\} + \{\bar{1}; 1\} \otimes \{1\} + \{\bar{1}; 1\} \otimes \{0\} = (\{\bar{2}; 2\} + \{\overline{1, 1}; 1, 1\} + 2\{\bar{1}; 1\} + 2\{0\}) \\
&\qquad + (\{\bar{1}; 1\} + \{0\}) + (\{0\})
\end{aligned} \tag{20}$$

so that there are seven couplings (or six arbitrary coefficients up to an overall normalization). Independent terms are most conveniently expressed in an $sl(n) + gl(1)$ basis, for which we introduce the generators $J^a{}_b$, $\hat{N}$, $\bar{Q}^a$, Q_b, $1 \leqslant a, b, c \leqslant n$ defined as follows:

$$\begin{aligned}
&J^a{}_b \equiv E^a{}_b - \frac{1}{n}\delta^a{}_b\hat{N} \qquad \hat{N} \equiv \langle E\rangle = E^c{}_c \\
&\left[J^a{}_b, J^c{}_d\right] = \delta_b{}^c J^a{}_b - \delta^a{}_d J_c{}^b \qquad \left[\hat{N}, J^a{}_b\right] = 0 \\
&\left[J^a{}_b, \bar{Q}^c\right] = \delta_b{}^c \bar{Q}^a - \frac{1}{n}\delta^a{}_b\bar{Q}^c \qquad [\hat{N}, \bar{Q}^c] = \bar{Q}^c \\
&\left[J^a{}_b, Q_c\right] = -\delta^a{}_c Q_b + \frac{1}{n}\delta^a{}_b Q_c \qquad [\hat{N}, Q_c] = -Q_c.
\end{aligned} \tag{21}$$

Finally the general anticommutator is written in terms of six arbitrary coefficients,

$$\begin{aligned}
&\{\bar{Q}^a, Q_b\} = (J^2)^a{}_b + \mathtt{a}\langle J^2\rangle\delta^a{}_b + (\mathtt{b}_1\hat{N} + \mathtt{b}_2)J^a{}_b + (\mathtt{c}_1\hat{N}^2 + \mathtt{c}_2\hat{N} + \mathtt{c})\delta^a{}_b \\
&\qquad \text{with} \quad \{\bar{Q}^a, \bar{Q}^b\} = 0 = \{Q_a, Q_b\}.
\end{aligned} \tag{22}$$

[12] Partitions labelling irreducible representations of the symplectic (see above) and orthogonal Lie algebras are denoted $\langle\lambda\rangle$, $[\lambda]$, respectively. (See appendix A.1 and [26]).

The coefficients $\mathtt{a}, \mathtt{b}_1, \mathtt{b}_2, \mathtt{c}_1, \mathtt{c}_2, \mathtt{c}$ are determined by demanding that (21), (22) above are consistent with the Jacobi identity,

$$[x, [y, z]] = [[x, y], z] + (-1)^{(x)\cdot(y)}[y, [x, z]] \tag{23}$$

for homogeneous $x, y, z \in L$ with $(x), (y) = \bar{0}$ or $\bar{1}$ being the $\mathbb{Z}_2$-grading of x, y, respectively. In view of the cyclic symmetry, there are four choices of three homogeneous elements, namely $\overline{000}, \overline{001}, \overline{011}, \overline{111}$, of which the first is simply the Jacobi identity for $L_{\bar{0}}$, while the second and third express the covariance of $L_{\bar{1}}$ and $\{L_{\bar{1}}, L_{\bar{1}}\}$ under $\mathrm{ad}_{L_{\bar{0}}}$ (which has been already built into (22)). By similar reasoning, the only nontrivial Jacobi identities involving three odd elements are

$$\begin{aligned}[\{\bar{Q}^a, Q_b\}, Q_c] &= [\bar{Q}^a, \{Q_b, Q_c\}] - [\{\bar{Q}^a, Q_c\}, Q_b] \equiv -[\{\bar{Q}^a, Q_c\}, Q_b] \\ &\text{and similarly} \quad [\{Q_a, \bar{Q}^b\}, \bar{Q}^c] = -[\{Q_a, \bar{Q}^c\}, \bar{Q}^b].\end{aligned} \tag{24}$$

Evaluating (24) explicitly using (21), (22) above we have

$$\begin{aligned}[\{\bar{Q}^a, Q_b\}, Q_c] &= \left[-(Q\cdot J)_b\delta^a{}_c - 2\mathtt{a}(Q\cdot J)_c\delta^a{}_b\right] + \left[-Q_bJ^a{}_c + \left(\mathtt{b}_1 - \frac{2}{n}\right)Q_bJ^a{}_c\right] \\ &\quad + \left[(-\mathtt{b}_1)Q_b\hat{N}\delta^a{}_c + \left(2\mathtt{c}_1 - \frac{\mathtt{b}_1}{n}\right)Q_c\hat{N}\delta^a{}_b\right] \\ &\quad + \left[-\left(\mathtt{b}_2 - \frac{1}{n}\right)Q_b\delta^a{}_c + \left(\frac{1}{n^2} - \frac{(n^2-1)}{n}\mathtt{a} - \frac{1}{n}\mathtt{b}_2 - \mathtt{c}_1 + \mathtt{c}_2\right)Q_c\delta^a{}_b\right]\end{aligned}$$

where $(Q\cdot J)_a \equiv Q_bJ^b{}_a$. Similarly

$$\begin{aligned}[\{\bar{Q}^a, Q_b\}, \bar{Q}^c] &= \left[(J\cdot\bar{Q})^a\delta^c{}_b + 2\mathtt{a}(J\cdot\bar{Q})^c\delta^a{}_c\right] + \left[\bar{Q}^aJ^c{}_b - \left(\mathtt{b}_1 - \frac{2}{n}\right)\bar{Q}^cJ^a{}_b\right] \\ &\quad + \left[\mathtt{b}_1\hat{N}\bar{Q}^a\delta^c{}_b - \left(2\mathtt{c}_1 - \frac{\mathtt{b}_1}{n}\right)\hat{N}\bar{Q}^c\delta^a{}_b\right] \\ &\quad + \left[\left(\mathtt{b}_2 - \frac{1}{n}\right)\bar{Q}^a\delta^c{}_b - \left(\frac{1}{n^2} - \frac{(n^2-1)}{n}\mathtt{a} - \frac{1}{n}\mathtt{b}_2 - \mathtt{c}_1 + \mathtt{c}_2\right)\bar{Q}^c\delta^a{}_b\right]\end{aligned}$$

with $(J\cdot\bar{Q})^a \equiv J^a{}_b\bar{Q}^b$. Imposing (24) yields

$$\begin{aligned}&\mathtt{a} = -\frac{1}{2} \qquad \mathtt{b}_1 = -\frac{n-2}{n} \\ &\mathtt{c}_1 = \frac{(n-1)(n-2)}{2n^2} \qquad \text{and} \qquad \frac{n-1}{n}\mathtt{b}_2 + \mathtt{c}_2 = -\frac{n-1}{2}\end{aligned} \tag{25}$$

with *no* restriction on $\mathtt{c}$. Finally it is always possible to absorb $\mathtt{b}_2$ by means of an appropriate shift $\hat{N} \to \hat{N}' \equiv \hat{N} + \mathrm{const}\mathbb{1}$, yielding the anticommutator for the quadratic algebra *uniquely* determined up to the central term,

$$\begin{aligned}\{\bar{Q}^a, Q_b\} &= (J^2)^a{}_b - \frac{1}{2}\langle J^2\rangle\delta^a{}_b - \frac{n-2}{n}\hat{N}J^a{}_b \\ &\quad + \delta^a{}_b\left[\frac{(n-1)(n-2)}{2n^2}\hat{N}^2 - \frac{(n-1)}{2}\hat{N}\right] + \mathtt{c}\delta^a{}_b\mathbb{1}.\end{aligned} \tag{26}$$

From (25) it can be seen that a more flexible set of algebraic defining relations emerges if, in addition to the anticommutation relations, covariant quadratic identities between the even and odd generators exist in the enveloping algebra. It is evident in any case from section 2 and the examples of section 3, that additional Serre-type relations can be expected for the consistency of the nonlinear algebras in general, and the present discussion is a case in point. Thus we impose the covariant conditions[13]

$$(J \cdot \bar{Q})^a = (\bar{\alpha}\hat{N} + \bar{\beta}\mathbb{1})\bar{Q}^a \qquad (Q \cdot J)_a = Q_a(\alpha\hat{N} + \beta\mathbb{1}) \tag{27}$$

for some constants $\alpha, \beta, \bar{\alpha}, \bar{\beta}$. This move releases the condition $\mathtt{a} = -\frac{1}{2}$ found above; the remaining coefficients are determined as in (25), with modified constraints

$$\begin{aligned} \mathtt{b}_1 &= -\frac{n-2}{n} \qquad \mathtt{c}_1 = \frac{(n-1)(n-2)}{2n^2} - (2\mathtt{a}+1)\alpha \\ \text{and} \qquad &\frac{n-1}{n}\mathtt{b}_2 + \mathtt{c}_2 = -\frac{n-1}{2} - (2\mathtt{a}+1)\beta \end{aligned} \tag{28}$$

and a corresponding three-parameter family $gl_2(n/\{1\}+\{\bar{1}\})^{\mathtt{a},\alpha,\beta}$ of quadratic algebras.

Further remarks concerning the quadratic algebras and the significance of the parameter α in constructing representations, are given in the conclusions (section 5 below). Note that the quadratic antisymmetric rank 3 algebra, constructed via fermion annihilation and creation operators in section 3, furnishes an example of this family, for the case $n = 4$ (with the identification of the rank 3 antisymmetric tensor representation with the contragredient of the defining representation for $sl(4)$). Define for $n = 4\bar{S}^i = \frac{1}{6}\epsilon^{ijk\ell}Q_{jk\ell}$, $S_i = \frac{1}{6}\epsilon_{ijk\ell}\bar{Q}^{jk\ell}$, then from (14) as shown in appendix A.4,

$$\{\bar{S}^i, S_j\} = (J^2)^a{}_b - \tfrac{1}{2}\langle J^2\rangle\delta^a{}_b - \tfrac{1}{2}\hat{N}J^a{}_b + \left(\tfrac{3}{16}\hat{N}^2 - \tfrac{3}{4}\hat{N} + 2\right)\delta^a{}_b$$

which agrees with (21), (25) for $n = 4$ $\left(a = -\frac{1}{2}, \mathtt{b}_1 = -\frac{1}{2}, \mathtt{c}_1 = \frac{3}{16}\right)$, together with $\mathtt{b}_2 = -1, \mathtt{c}_2 = -\frac{3}{4}$. Details of the calculation, together with further consideration of the covariant identities (27), are provided in appendix A.4.

As a final development we consider some aspects of the representation theory of the nonlinear super-$gl(n)$ algebras, as exemplified by the quadratic family $gl_2(n/\{1\}+\{\bar{1}\})^{\mathtt{a},\alpha,\beta}$. Parallels with the representation theory of classical superalgebras are brought out by the construction of induced modules, and consideration of their (a)typicality conditions. Following Kac's construction for type I superalgebras [46], and in particular, for $sl(n/1)$, an analogous induced module $\bar{V}\{\lambda\}$ for $gl_2(n/\{1\}+\{\bar{1}\})^{\mathtt{a},\alpha,\beta}$ may be constructed via the choice of a Borel superalgebra B_+, its associated enveloping superalgebra U_+ and an arbitrary[14] L_0-module $\bar{V}_0\{\lambda\}$ extended trivially to B_+. Then with the help of the Poincaré–Birkoff–Witt theorem we have as usual

$$\bar{V}\{\lambda\} \simeq \bigwedge(L_-) \otimes \bar{V}_0\{\lambda\}$$

or explicitly,

$$\bar{V}\{\lambda\} = \sum_{k=0}^{n} \sum_{a_1,a_2,\ldots,a_k} Q_{a_1}Q_{a_2}\cdots Q_{a_k} \otimes \bar{V}_0\{\lambda\}.$$

Introducing the highest weight vector v^+, of particular interest is the vector $Q_n \otimes v^+$, which (as it commutes with even raising operators $E^a{}_b$ for $1 \leqslant a < b \leqslant n$), will again be a B_+-highest weight vector, and moreover will cyclically generate an indecomposable submodule of $\bar{V}\{\lambda\}$ of highest weight *different* from λ, if $\bar{Q}^n Q_n \otimes v^+ = 0$ (since $\bar{Q}^a Q_n \otimes_{U_+} v^+ = \{\bar{Q}^a, Q_n\}v^+_\lambda = 0$ for $a < n$). Thus a necessary condition for typicality is the nonvanishing of the eigenvalue of $\{\bar{Q}^n, Q_n\}$ on v^+. Similar considerations in fact apply to the hierarchy of vectors $Q_n \otimes v^+, Q_{n-1}Q_n \otimes v^+, \ldots$ (see [46]).

[13] Further details of the generalization of these Serre-type relations to the cases $gl_2(n/\{1^3\}+\{\overline{1^3}\})$ and $gl_2(n/\{3\}+\{\bar{3}\})$, as well as the present case, are given in appendix A.4.

[14] Note that here, in contrast to the previous notation, $\bar{V}\{\lambda\}$ is a Kac module based on an *arbitrary* L_0-highest weight, but for the *fixed* super-$gl(n)$ algebra $gl_2(n/\{1\}+\{\bar{1}\})$.

The connection with standard lexicographical (partition) labelling is simplest if the anticommutation relations (26) are re-written directly in terms of the standard $gl(n)$-generators $E^a{}_b$ (see (21)), yielding

$$\{\bar{Q}^a, Q_b\} = (E^2)^a{}_b - \hat{N}E^a{}_b - \tfrac{1}{2}\delta^a{}_b[\langle E^2\rangle - \hat{N}(\hat{N} - n + 1)] + \mathtt{c}\delta^a{}_b\mathbb{1}. \qquad (29)$$

With the highest weight labels $\lambda_1, \lambda_2, \ldots, \lambda_n$ of $gl(n)$ (eigenvalues of $E^1{}_1, E^2{}_2, \ldots, E^n{}_n$), for a dominant integral weight λ such that $\lambda_a - \lambda_b \in \mathbb{Z}^+$ for $a > b$, and denoting the (eigenvalues of the) first and second degree Casimir operators by Λ, $\mathtt{C}$, we have, for example, for the eigenvalue a_n of $\{\bar{Q}^n, Q_n\}$ by direct computation,

$$a_n = \lambda_n(\lambda_n - \Lambda) - \tfrac{1}{2}\mathtt{C} + \tfrac{1}{2}\Lambda(\Lambda - n + 1) \qquad (30)$$

where the usual eigenvalues are understood:

$$\Lambda = \sum_{a=1}^{n} \lambda_a \qquad \mathtt{C} = \sum_{a=1}^{n} \lambda_a(\lambda_a + n + 1 - 2a). \qquad (31)$$

5. Conclusions

In this paper we have made a preliminary investigation of a large class of 'polynomial super-$gl(n)$ algebras'. These mimic the classical simple Lie superalgebras $sl(n/1) \sim A(n-1, 0)$ in possessing an even part $gl(n) \simeq sl(n) + gl(1)$, however, with odd generators in an *arbitrary* representation $\{\lambda\}$ of $gl(n)$ and its contragredient, provided that the anticommutator of odd generators closes on a *polynomial* (of degree >1) in the even generators of $gl(n)$. A general discussion of admissible structure constants (section 2) was exemplified by concrete fermionic oscillator constructions in specific cases (section 3), and for $gl_2(n/\{1\} + \{\bar{1}\})^{\mathtt{a},\alpha,\beta}$ (quadratic superalgebras with odd generators in the defining n-dimensional representation of $gl(n)$) a unique set of structure constants presented (section 4), together with the elements of the construction of finite-dimensional irreducible representations.

As discussed in section 1, our polynomial superalgebras are allied to classes of nonlinear algebras already studied in various physical settings. For example, polynomial deformations of $sl(2)$ of degree Δ including the so-called Higgs [5] case ($\Delta = 2$) have been studied by Beckers [6]; nonlinear extensions of supersymmetric quantum mechanics have been identified in [9]. Recently a broad class of 'polynomial Lie algebras' has been found [10] in the context of anharmonic interactions in second quantized descriptions of many-body systems. In [10] the relationship of such models to integrable systems is studied. In [11], various examples of polynomial Lie algebras were identified via their bosonic oscillator realizations. There the abstract status of such nonlinear algebras was not taken up to the extent of systematic detailed study of admissible Jacobi identities and structure constants, as in the present work for the super case. However, many interesting variants of nonlinear polynomial algebras were obtained in oscillator constructions, including a classification of the three-dimensional case (polynomial deformations of $su(2)$ and $su(1,1)$), and results on representations. Analogous considerations for the $su(1,1)$ case are implicit also in recent work on 'K-quantum' ladder operators and associated coherent and squeezed states [12]. In terms of classical and quantum dynamical systems, in [7] various quadratic Poisson–Lie symmetry algebras have been investigated (together with quantum versions) in connection with various models of potentials admitting separation of variables. Similarly the Askey–Wilson three-dimensional quadratic Poisson–Lie symmetry algebra (see for example [8] and references therein) plays a key role in nonlinear integrable systems. Finally, as noted in section 1, the super-$gl(n)$ algebras are structurally closely related to the nonlinear W-algebras and W-superalgebras derived from the hamiltonian reduction on coadjoint Lie–Poisson manifolds which have been classified and studied in recent work [1–3]. Note that several examples of the latter which have been presented [4] can readily be transcribed into the covariant tensor notation used in the present paper. For example, the W-algebra defined by the regular $sl(2)$ embedding within $sl(4)$, $\mathbf{4} \to \mathbf{2}_0 + \mathbf{1}_1 + \mathbf{1}_{-1}$ possesses an undeformed subalgebra $sl(2) + gl(1)$ with generators E, F, H and U, together with a pair of doublets $\bar{G}_{\pm\frac{1}{2}}$, $G_{\pm\frac{1}{2}}$ with equal and opposite charge under U. Defining (see (A.8))

$$\begin{pmatrix} E^1{}_1 & E^1{}_2 \\ E^2{}_1 & E^2{}_2 \end{pmatrix} = -\begin{pmatrix} U+H & F \\ E & U-H \end{pmatrix}$$

the quadratic closure relations (8.4) of [4] read

$$[\bar{G}^a, G_b] = a(E^2)^a{}_b + bE^a{}_b + c\delta^a{}_b \mathbb{1}$$

in complete analogy with (26). The study of the relationship between the present polynomial superalgebras, and Hamiltonian reduction, is thus likely to shed light on their geometrical interpretation, in relation to their applications in quantum field theory (see below). In particular, the meaning of 'finite' symmetry transformations associated with deformed algebras requires elaboration.

In recent work on non-perturbative aspects of gauge field theories, the structure of the observable algebra has been investigated within the Hamiltonian formulation on a finite lattice (see [15–19]), and has led to the need to study polynomial Lie superalgebras as a natural part of the algebraic structure. For completeness we briefly mention the context of these investigations, in order to explain the connection with the present work.

By definition, the observable algebra is the algebra of gauge invariant elements built from field operators, satisfying the Gauss law. In a first step, this algebra can be explicitly characterized in terms of generators and relations. Next, it has to be endowed with an appropriate functional analytic structure and, finally, one has to classify its irreducible representations. For quantum electrodynamics this programme has been implemented completely. It turns out that the observable algebra naturally decomposes into a bosonic part, which is isomorphic to a Heisenberg algebra of canonical commutation relations, and a matter field part. For the case of spinor electrodynamics [15], the matter field part turns out to be generated by the Lie algebra $u(2N)$, with N denoting the number of lattice sites. For scalar electrodynamics [16], it is generated by $u(N, N)$. In both cases, irreducible representations are labelled by the total electric charge, yielding a decomposition of the physical Hilbert space into charge superselection sectors.

In the case of quantum chromodynamics (QCD), a full analysis of the structure of the observable algebra is much more complicated (see [17–19]). Here, quark matter fields are introduced as canonical fermionic operators $\psi^{*a}{}_A(\mathbf{x})$, $\psi_a{}^A(\mathbf{y})$, with $a = 1, 2, \ldots, s$ spin $_{\alpha}, ^{\dot{\alpha}} = 1, 2$, and flavour indices, while $A, B = 1, 2, 3$ are $su(3)$ colour indices and $\mathbf{x}, \mathbf{y} = 1, 2, \ldots, N$ are lattice sites with $N = L^D$ for a cubic lattice in spatial dimension D. Writing $a\mathbf{x} \equiv i$, $b\mathbf{y} \equiv j, \ldots$, natural colour invariant operators built from quark fields are then

$$E^i_{\gamma,j} := \psi^{*i}{}_A U^A_{\gamma B} \psi_j{}^B \tag{32}$$

$$W_{\alpha\beta\gamma,(ijk)} := \tfrac{1}{6}\epsilon_{ABC} U^A_{\alpha D} U^B_{\beta E} U^C_{\gamma F} \psi_i{}^D \psi_j{}^E \psi_k{}^F \tag{33}$$

with $U^A_{\gamma B}$ denoting the parallel transporter along γ, built from the gluonic gauge fields. In formula (32), γ denotes an arbitrary curve from $\mathbf{x}$ to $\mathbf{y}$, whereas in (33) α, β and γ are arbitrary curves starting at some reference point $\mathbf{t}$ and ending at $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$, respectively. The invariant operators $E^i_{\gamma,j}$ and $W_{\alpha\beta\gamma,(ijk)}$ represent hadronic matter of mesonic and baryonic type. These elements, together with a set of purely gluonic invariants [18, 19] constitute a set of generators.

This set, however, is highly redundant. There is a number of non-trivial relations between generators, inherited from the canonical (anti)commutation relations and from the local Gauss laws. A complete discussion of the observable algebra as an abstract algebra in terms of generators and defining relations will be presented in [19].

A method for solving a large part of the relations consists in choosing a lattice tree. Suppose that a tree has been fixed. Restricting ourselves then to invariant operators (32) and (33), with α, β and γ being the unique on-tree paths and the reference point being the lattice root, these operators coincide with generators $\left(E^i{}_j, W_{(ijk)}, \bar{W}^{(ijk)}\right)$ of the algebra $gl_2(n/\{3\}+\{\bar{3}\})$ discussed in section 3. A slightly delicate gauge orbit analysis, together with some further tree techniques, enables one to further reduce the number of generators, leading to the algebra $gl_2(n/\{1^3\}+\{\overline{1^3}\})$, defined in section 3, with $gl(n)$-generators given by formula (13) and odd generators given by (12). These generators still inherit some relations, but now the algebra has become tractable. It can be shown that it is isomorphic to the universal enveloping algebra of $sl(N/1)$, factorized by a certain ideal defined in terms of relations on Casimir operators of a certain ordinary Lie superalgebra. Moreover, it is then easy to prove that irreducible representations of this algebra are labelled by the global colour charge (triality), built from the local colour charge densities carried by the quark field [18, 19].

One aspect of the structure of 'composite operators' such as the colour singlets $\bar{W}^{(ijk)}$ which emerges from the nonlinear algebra perspective has potentially wide applicability. Consider the reduction of degeneracy associated with states obtained by acting with monomials in $\bar{W}^{(ijk)}$ on the vacuum, relative to what would be obtained if the $\bar{W}^{(ijk)}$, $W_{(pqr)}$ were elementary fermions (see appendix A.3). In the context of the representation theory of the polynomial super-$gl(n)$ algebras, the reduced representation content is a natural consequence of the Fock space realization being generically an *atypical* representation.

'Gauge invariance' is often handled by covariant BRST methods, which circumvent noncovariant Hamiltonian approaches. However, as a purely algebraic problem, the Gauss' law constraint can be also introduced via cohomology in Hamiltonian BRST formulations. To this end the equicovariant formalism of [47] should be noted, wherein baryonic colour singlets such as (33) are naturally identified as nontrivial cocycles at nonzero (but triality zero) ghost number.

Applications of polynomial superalgebras in quantum field theory relate to spacetime supersymmetry. In Hamiltonian lattice QCD, the quadratic superalgebra is a *bona fide* fermion–boson supersymmetry between baryon and meson states[15]. There is a possibility that 'no-go' theorems for the combination of internal and spacetime symmetries—circumvented for supersymmetry to the extent of allowing N-extended Poincaré Fermi–Bose supersymmetries [49]—can be further relaxed for nonlinear supersymmetries. Also, in the case $n = 4$, appropriate real forms of $gl_2(4/\{1\}+\{\bar{1}\})$ may allow various six-dimensional realizations, or even new types of conformal supersymmetry in four dimensions ($su(2,2) \simeq so(4,2)$). Such superalgebras therefore add further to the resource of available generalized supersymmetries in diverse dimensions, following, for example, [50], or recently [51] for new higher dimensional superstring and supermembrane algebras[16].

In relation to conformal and other spacetime symmetries, it should be noted that antecedents of our polynomial algebras and superalgebras have been encountered before in connection with representation theory. For example Barut and Bohm [13] identify certain so-called special 'representation relations' which are *anticommutation* relations between the

[15] Free field current algebras of this type within relativistic spin-flavour symmetry models were considered by Delbourgo *et al* [48].

[16] Parafermionic generalizations of Poincaré supersymmetry have also been considered; see [52].

standard generators P_μ and K_μ and the Lorentz and dilatation generators $J_{\mu\nu}$ and D of the form

$$\{P_\mu, K_\mu\} = (a\mathbb{1} + bD)\eta_{\mu\nu} + \left\{J_\mu{}^\sigma, J_{\sigma\nu}\right\}$$

which are obtained in certain classes of representation of the four-dimensional conformal algebra $so(4,2)$. Similarly Bracken [14] in studying algebraic properties of the Gel'fand–Yaglom matrices Γ_μ in higher spin wave equations introduced analogous algebraic relations (but without the D term) for $\{\Gamma_\mu, \Gamma_\nu\}$ as a generalization of the Dirac algebra[17]. From the perspective of the present work, the above relations provide instances of the structure constants of quadratic superalgebras, in this case of the $so_2(3, 1/[1])$ type, where the 'odd' generators transform as vector operators. In contrast to the original contexts, however, any *Lie algebra* relations which such vector operators happen to satisfy are now relegated to the status of specific 'representation relations', with the *anticommutation* relations regarded as primary.

In concrete applications the general question of a representation theory for the new polynomial superalgebras in their own right arises. In particular the existence of a tensor category associated with coproduct and Hopf structures needs further investigation, and the role of 'deformations' needs clarification. In the lattice QCD case, there is also the possibility that interesting structures may emerge only as *local* entities in the thermodynamic ($N \to \infty$) limit. Similarly, in the spacetime supersymmetry case, the appropriate context may be contraction limits of radii of additional dimensions, or orbifold parameters such as brane tensions.

In sections 3 and 4 above, examples of polynomial superalgebras were found in which the Jacobi identities are underwritten by additional covariant Serre-type relations in the enveloping algebra. Such identities are likely to be the rule rather than the exception for the nonlinear case, and may profoundly affect representations. A precedent for such phenomena exists in the so-called 'multiplet shortening' for massless supermultiplets in N-extended supersymmetries [53]. For appropriate kinematical conditions, the (spinor) supercharges are subjected to covariant constraints of the form $P_\mu \gamma^\mu{}_\alpha{}^\beta Q_\beta = 0$. In the usual Wigner induced representation method, this situation is easily handled as the representations of the Abelian translation part are one dimensional, and moreover P_μ is taken in a standard Lorentz frame. In the $gl_2(n/\{1\} + \{\bar{1}\})$ and other cases, the constraints are also of covariant form, $E^a{}_b \bar{Q}^b \propto \bar{Q}^a$, $Q_a E^a{}_b \propto Q_b$, but of course the multiplier $E^a{}_b$ is non-Abelian (for $n = 4$, if $sl(4)$ can be identified with the real form $so(4,2)$, with $\bar{Q}$, Q identified with the spinor representation of the latter, the above $\gamma \cdot P$ term would certainly appear as one contribution to the constraint). In this connection, methods developed in recent work [54, 32, 55] for the explicit construction of all (including atypical) finite-dimensional irreducible modules of type I Lie superalgebras can be adapted, at least for a class of representations of polynomial Lie superalgebras. In the ordinary Lie superalgebra case [54], the polynomial identities satisfied by the $gl(n)$ Gel'fand generators play a crucial role in deriving $gl(n)$ branching rules at each 'floor' of the Kac module $\left[\sum_{k=0}^{n} \oplus \bigwedge^k (L_-)\right] \otimes_{U_+} V(\Lambda)$. The same method can be generalized to the polynomial super-$gl(n)$ case; the implications of the additional covariant Serre-type relations as constraints on the structure of the induced modules can be ascertained in precisely this framework[18]. Full analysis along these lines, especially for the polynomial superalgebras related to Hamiltonian lattice QCD, is the subject of future development.

[17] The Dirac algebra itself is of course a 'polynomial superalgebra' of degree zero!

[18] In the tensor operator language [65], relations such as $E^a{}_b \bar{Q}^b = q\bar{Q}^a$, $Q_a E^a{}_b = Q_b q$ in the $gl_2(n/\{1\} + \{\bar{1}\})$ case (and generalizations to $gl_k(n/\{\lambda\} + \{\bar{\lambda}\})$, see appendix A.4) imply that $\bar{Q}$, Q are shift operators for certain states in the $gl(n)$ modules $\{\lambda\}$, $\{\bar{\lambda}\}$ depending on the eigenvalue q.

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Appendix

A.1. Partition labelling for irreducible representations of $gl(n)$ and structure constants of polynomial super-$gl(n)$ algebras

The use of tensor notation has been formalized by Weyl [56], Hamermesh [57] and others in treatises on the relation between partitions and irreducible finite-dimensional representations of the group $GL(n)$. A central role in the character theory is played by the Schur functions, especially as developed by Littlewood [28] for the unitary group $U(n)$ and subgroups $SU(n)$, $O(n)$ and $Sp(n)$. Many aspects of the theory have been developed further for arbitrary semisimple (including exceptional) Lie groups, culminating in the extensive tabulations of [26].[19] Most of the algorithms have been implemented in the group theory package ©SCHUR [21]. Here we outline the necessary elements of the formalism for the case of finite-dimensional irreducible representations of $GL(n)$ required for the computation of structure constants (section 1) and branching rules (section 3 and appendix A.2).

Finite-dimensional irreducible representations of $GL(n)$ (corresponding to dominant integral highest weight modules of the simple complex Lie algebra $sl(n)$) are labelled[20] by partitions $\{\lambda\} = \{\lambda_1, \lambda_2, \ldots, \lambda_\ell\}$ for non-negative integers $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_\ell \geqslant 0$, where $\ell \leqslant n$ is the number of parts of $\{\lambda\}$, and $\{\lambda\}$ has weight or rank $|\lambda| = \lambda_1 + \lambda_2 + \cdots + \lambda_\ell$; λ is a partition of $|\lambda|$, $\lambda \vdash |\lambda|$. $\{\lambda\}$ is represented graphically by a Young tableau, which is an array of left-top justified rows of boxes, of lengths corresponding to the parts of $\{\lambda\}$.

Tensor or Kronecker products of the modules $\{\lambda\}$ and $\{\mu\}$ are evaluated by the celebrated Littlewood–Richardson rule, which gives the resolution of the product of the corresponding characters (referred to as *outer* multiplication of Schur S-functions),

$$\{\lambda\} \cdot \{\mu\} = \sum_{\nu} C_{\lambda\mu}{}^{\nu}\{\nu\} \tag{A.1}$$

[19] Supersymmetric partition labelling and Young diagrams have also been introduced for representations of classical superalgebras; see for example [30, 31, 58]. For an example of the organizing power of group methods in tensor notation (applied to higher order heat kernel coefficients in curved space backgrounds) see, for example, [59]. See [60, 61] for applications of supersymmetric Schur functions to infinite-dimensional algebras.

[20] In Littlewood's nomenclature [28, 26] the symbols $\{\lambda\}$, $[\lambda]$, $\langle\lambda\rangle$ pertain to $GL(n)$, $O(n)$ and $Sp(n)$, respectively.

via the Littlewood–Richardson coefficients $C_{\lambda\mu}{}^{\nu}$, where $|\nu| = |\lambda| + |\mu|$. Dually related is the definition of S-function *skew*,

$$\{\nu/\lambda\} = \sum_{\mu} C_{\lambda\mu}{}^{\nu}\{\mu\} \tag{A.2}$$

where the sum is over all $\{\mu\}$ such that $\{\lambda\}\cdot\{\mu\} \ni \{\nu\}$, with the coefficient in the skew being given by the appropriate product multiplicity.

Various other S-function operations are needed for the manipulation of group representations and branching rules. Important is the *inner* S-function product, defined for partitions of equal rank, $|\lambda| = |\mu|$,

$$\{\lambda\}\circ\{\mu\} = \sum_{\nu} \Gamma_{\lambda\mu}{}^{\nu}\{\nu\}$$

where $|\nu| = |\lambda| = |\mu|$, which gives the resolution of the corresponding product of characters (tensor product of representations) of the symmetric group. Finally there is the S-function *plethysm* $\{\lambda\}\otimes\{\mu\}$ which resolves the tensor product of the irreducible representation $\{\lambda\}$ with itself, $|\mu|$ times, into its projection of symmetry type μ, with respect to the action of the permutation group $S_{|\mu|}$ on the factor spaces.

The manifest advantage of partition notation, even with its more complicated extension to orthogonal and symplectic groups and even exceptional groups (see [26]), is its general feature of being *rank-independent* for groups of large enough dimension, and generic representations. Any corrections for specific groups (such as say $SU(3)$ or $SO(10)$), are done by means of group-dependent *modification rules* which rule out illegal partitions resulting from general algorithms, and in some cases relate characters specified by non-standard partitions to standard ones, up to signs (which must be collected at the end of a calculation).

The major modification rule for $GL(n)$ is simply that partitions of more than n parts (diagrams with more than n rows) vanish identically. In addition, for $SL(n)$, columns of length n can be deleted. When dealing with both covariant and contravariant representations of $GL(n)$, it is natural to introduce a more flexible mixed or composite partition notation $\{\bar{\lambda};\mu\}$ which represents a tensor of mixed contravariant and covariant rank $|\lambda|, |\mu|$ respectively, but for which all tensor contractions between upper and lower indices vanish [27]. Standard partitions of this type have *total* number of parts (rows of λ and μ) at most n, and are equivalent to canonical pure covariant or pure contravariant irreducible representations up to powers of the one-dimensional alternating character (the determinant). Non-standard partitions of mixed type are either zero (for example if the number of parts is identically $n+1$) or modify in specific ways to standard tableaux. We do not require the general rules (see [26] and [27]), which have been implemented in ©SCHUR [21].

The most obvious application of the composite notation is in handling the n^2-dimensional adjoint representation of $GL(n)$. Technically this is isomorphic to the tensor product of the defining representation $\{1\}$ and its contragredient $\{\bar{1}\}$, written as

$$\{\bar{1}\}\cdot\{1\} = \{\bar{1};1\} + \{0\}$$

with $\{\bar{1};1\}$ representing the traceless part (the adjoint representation of $SL(n)$), and $\{0\}$ the trace (the linear Casimir invariant). See section 4, (21) for the explicit reduction. For the familiar case of $SU(3)$, the composite notation is convenient, in that all finite-dimensional irreducible representations can be specified by symmetrized tensors of mixed type. For example, $\{\bar{2}\} \equiv \{2^2\} \equiv \bar{\mathbf{6}}$, $\{\bar{2};2\} \equiv \{4,2\} \equiv \mathbf{27}$, and the product

$$\{\bar{2}\}\cdot\{2\} = \{\bar{2};2\} + \{\bar{1};1\} + \{0\}$$

corresponds to the reduction

$$\bar{\mathbf{6}}\times\mathbf{6} = \mathbf{27}+\mathbf{8}+\mathbf{1}.$$

We now formalize the above and other required tensor product and branching rules required in sections 2 and 3. For the tensor product of contravariant and covariant irreducible representations we have

$$\{\bar{\lambda}\} \cdot \{\mu\} = \sum_{\alpha} \{\overline{\lambda/\alpha}; \mu/\alpha\} \tag{A.3}$$

where the skew is performed with respect to all legal $\{\alpha\}$, and the expansion of the skew via (A.2) is done distributively.

For the decomposition of an irreducible representation of $GL(pq)$ with respect to $GL(p) \times GL(q)$ we have

$$\{\lambda\} \downarrow \sum_{\sigma \vdash |\lambda|} \{\sigma\} \times \{\sigma \circ \lambda\}. \tag{A.4}$$

In particular, if $\{\lambda\} = \{\ell\}$, then $\{\sigma \circ \lambda\} = \{\sigma\}$ as $\{\ell\}$ labels the trivial representation of S_ℓ. Alternatively if $\{\lambda\} = \{1^\ell\}$, then $\{\sigma \circ \lambda\} = \{\sigma'\}$, the partition transpose to $\{\sigma'\}$ (with rows and columns interchanged) as $\{1^\ell\}$ is the one-dimensional alternating character of S_ℓ.

The rules for symmetric function plethysm have been developed by Littlewood (see appendix to [28]) and others; see for example [62–64]. The algorithm for plethysm is implemented in the group theory package ©SCHUR [21]. However, for the applications needed in section sections 2 and 4 above, the following rules suffice for the evaluation of low-rank cases[21]:

$$\{\bar{1}\} \cdot \{1\} \otimes \{\ell\} = \sum_{\sigma \vdash \ell} \{\bar{\sigma}\} \cdot \{\sigma\} = \sum_{\sigma \vdash \ell} \sum_{\alpha} \{\overline{\sigma/\alpha}; \sigma/\alpha\} \tag{A.5}$$

whereas for the irreducible part

$$\begin{aligned}
\{\bar{1}; 1\} \otimes \{\ell\} &\uparrow (\{\bar{1}\} \cdot \{1\} - \{0\}) \otimes \{\ell\} \\
&= \sum_{m} (-1)^m (\{\bar{1}\} \cdot \{1\}) \otimes \{\ell - m\} \cdot \{0\} \otimes \{m\} \\
&= \sum_{m} (-1)^m \sum_{\sigma \vdash \ell - m} \{\bar{\sigma}\} \cdot \{\sigma\} \downarrow \sum_{m} (-1)^m \sum_{\sigma \vdash \ell - m} \sum_{\alpha} \{\overline{\sigma/\alpha}; \sigma/\alpha\}.
\end{aligned} \tag{A.6}$$

From (A.5) it is evident in comparison with (2) and (A.1) that

$$n^k{}_{\mu\nu} = \sum_{\alpha \vdash k, \gamma} C_{\gamma\mu}{}^{\alpha} C_{\gamma\nu}{}^{\alpha} \qquad n^\lambda{}_{\mu\nu} = \sum_{\gamma} C_{\gamma\mu}{}^{\lambda} C_{\gamma\nu}{}^{\lambda} \tag{A.7}$$

from which $n^k{}_{\mu\nu} \geqslant n^\lambda{}_{\mu\nu}$ provided $k \geqslant \ell$.

A.2. Generalized Gel'fand notation for $gl(n)$ defining relations and structure constants of polynomial super-$gl(n)$ algebras

We reiterate briefly here for the case of $gl(n)$, a framework for the theory of characteristic identities for semisimple Lie algebras, which puts the Gel'fand notation for the defining relations in a broader context, and has been used in an essential way for the resolution of the structure of atypical modules of type I classical superalgebras.

Taken as a whole, within a certain (irreducible) representation, the array of Gel'fand generators $E^a{}_b$, $1 \leqslant a, b \leqslant n$ can be regarded as an invariant $E \in \pi(gl(n)) \otimes \mathrm{End}(\mathbb{C}^n)$, where $\mathbb{C}^n \equiv V\{1\}$ is the irreducible n-dimensional defining representation, and $\pi : gl(n) \to \mathrm{End}(\mathcal{V})$

[21] Note that the alternating signs in the final expression are typical of the outcome of Schur function manipulations, where a final positive sum of characters is only apparent after modification rules and cancellations have been accounted for.

is an algebra homomorphism for some $gl(n)$-module $\mathcal{V}$. The corresponding degree k invariants E^k within $\pi(U(gl(n))) \otimes \mathrm{End}(V\{1\})$, are nothing but the above matrix powers $(E^k)^a{}_b$ of the array of Gel'fand generators; the traces $\langle E^k \rangle$ are of course the standard Casimir operators of $gl(n)$.

This construction generalizes to an invariant $\mathcal{E} \in \pi(gl(n)) \otimes \mathrm{End}(V\{\lambda\})$ for an arbitrary irreducible representation $\{\lambda\}$. The matrix elements with respect to a basis of $V\{\lambda\}$ of $\mathcal{E}^k \in \pi(U(gl(n))) \otimes \mathrm{End}(V\{\lambda\})$ will provide precisely the leading unique degree k coupling for the polynomial superalgebra, and moreover related partial traces enable the remaining lower degree couplings to be enumerated in accord with the above counting schemes.

Let $\mathtt{C}$ be the second-order Casimir invariant (see (31) above). The general definition of $\mathcal{E}$ is

$$\mathcal{E} = \frac{1}{2}(\pi \otimes \mathbb{1}) \circ (\Delta(\mathtt{C}) - \mathtt{C} \otimes \mathbb{1} - \mathbb{1} \otimes \mathtt{C}). \tag{A.8}$$

Finally, if $e_a \otimes e_b \otimes \cdots$ are an (appropriately symmetrized) basis for $V\{\lambda\}$, then the set of $gl(n)$ generators in generalized Gel'fand notation is defined by the the matrix elements

$$\mathcal{E}^{abc\ldots}{}_{pqr\ldots} = (e_a \otimes e_b \otimes \cdots, \mathcal{E} e_p \otimes e_q \otimes \cdots) \tag{A.9}$$

as operators in $\mathrm{End}(\mathcal{V})$.[22]

A.3. Decomposable representations of $gl_2(n/\{3\} + \{\bar{3}\})$

The examples of oscillator realizations which we have considered not only provide the defining relations of various types of polynomial super-$gl(n)$ algebras, but also furnish examples of representations. In the fermionic case there are thus finite-dimensional, generally decomposable, super-$gl(n)$ representations in Fock space, via the usual action, and in the associated Clifford algebra, via the adjoint action. In this appendix we study the case $gl_2(n/\{3\} + \{\bar{3}\})$. The relevant state space and adjoint operators are classified in the general case, and for concreteness the results for the simplest cases $n = 1, n = 2$ are listed explicitly.

As discussed in section 3 above, within the $m = 6n$-dimensional Clifford algebra generated by the fermionic creation and annihilation operators a^{iA}, a_{jB}, $gl_2(n/\{3\} + \{\bar{3}\})$ is generated by $\bar{W}^{(ijk)}$, $W_{(pqr)}$, and $E^i{}_j$ realized as colour traces. The total Fock space is as usual an irreducible spinor representation of $so(6n) \supset gl(3n)$, and the Clifford algebra is embedded naturally as all endomorphisms on this space; taking account of the grading, therefore, we have $gl(n/\{3\} + \{\bar{3}\})_2 \subset C\ell(6n) \subset gl(2^{3n-1}/2^{3n-1})$.

Given normal ordering conventions, and the usual construction of states using creation modes applied to the vacuum state, the problem of classifying colour singlet states in Fock space is in fact a sub-case of that of identifying all colour-singlet operators. In general, we consider the reduction of the tensor representation of $gl(3n)$, corresponding to the product

$$X^{i_1A_1i_2A_2\ldots i_KA_K}{}_{p_1B_1p_2B_2\ldots p_LB_L} = a^{i_1A_1}a^{i_2A_2}\ldots a^{i_KA_K}a_{p_1B_1}a_{p_2B_2}\ldots a_{p_LB_L}$$

[22] The above formalism has been used to derive polynomial characteristic identities for generators of Lie algebras and superalgebras. See [65] for original works (see also [66]), and [67, 58] for extensions to superalgebras. For abstract approaches see [68–70]. For the relationship of the construction to Casimir invariants of arbitrary degree see [38, 39]. For the role of Yangians in relation to Laplace operators for Lie algebras and noncommutative characteristic polynomials see [40]. For the relation to the Goddard–Kent–Olive construction [71] in the affine case see [72].

to $gl(3) + gl(n)$, with the identification of $sl(3)$ singlets (colour invariant *states* correspond to the $L = 0$ case)[23]. This is a standard group reduction problem[24], and can be efficiently handled via the extended partition labelling (see appendix A.1 above, and [26]), resulting in

$$gl(3n) \downarrow gl(3) + gl(n) \quad \{\overline{1^K}\} \cdot \{1^L\} \downarrow \sum_{\rho \vdash K, \sigma \vdash L} \{\overline{\rho'}\} \cdot \{\sigma'\} \times \{\bar{\rho}\} \cdot \{\sigma\} \tag{A.10}$$

where $0 \leqslant K, L \leqslant 3n$, and $\{\rho'\}, \{\sigma'\}$ are the transpose partitions to $\{\rho'\}, \{\sigma'\}$ such that in the permutation groups S_K, S_L we have $\{\rho' \circ \rho\} \ni \{1^K\}$, and similarly $\{\sigma' \circ \sigma\} \ni \{1^L\}$ (see appendix A.1, and [26]). Using the usual restrictions that partitions represent nonzero characters of $gl(n)$ provided they have at most n rows, it can be seen from (A.10) that both ρ and σ must fall within a rectangular envelope of standard shape $3 \times n$. Finally, the right-hand side of (A.10) should be reduced with respect to $sl(3)$, which entails the further modification rule that columns of ρ', σ' of length 3 can be removed. Thus for $K = 0$, the branching rule gives an $sl(3)$ singlet provided $\{\sigma\} = \{3^r\}, r = 0, 1, 2, \ldots, n$ ('multi-baryons'), and similarly for $L = 0$ we have $\{\bar{\rho}\} = \{3^r\}, r = 0, 1, 2, \ldots, n$ ('multi anti-baryons', respectively). For both $K, L \neq 0$, we count $sl(3)$ colour singlets within $\{\overline{\rho'}\} \cdot \{\sigma'\}$ for legal partitions ρ', σ' within the $n \times 3$ rectangle. The relevant branching rule is (see appendix A.1 and [26])

$$\{\overline{\rho'}\} \cdot \{\sigma'\} \downarrow \sum_{\alpha} \{\overline{\rho'/\alpha}; \sigma'/\alpha\}$$

where the sum is over all partitions α whose skew with both ρ', σ' is nonvanishing. Obviously, the sum contains a singlet if and only if $\rho = \sigma$ (skewing by $\alpha = \rho = \sigma$ leads to the trivial representation). For $K = L$ this immediately gives a classification of all 'meson' colour singlets within $gl(n)$, classified by the reduction of $\{\bar{\sigma}\} \cdot \{\sigma\}$ for σ within the standard $3 \times n$ rectangular envelope. For $K > L$ or $K < L$ the same theorem applies, but the equality of ρ and σ may arise as a result of modification by dropping columns of length 3. This provides a classification of all 'exotic baryons and anti-baryons', according to $gl(n)$ multiplets arising in the reduction of $\{\overline{3^r \wr \lambda}\} \cdot \{3^s \wr \lambda\}$, where $r \neq s$, $0 \leqslant r, s \leqslant n$, where λ lies within a truncated $2 \times (n - t)$ rectangular envelope, where $t = \max(s, t)$ and the notation $\{3^r \wr \lambda\}$ indicates that λ is appended *below* the relevant rectangular block of depth r. This classification in fact includes the previous $K = 0$ and $L = 0$ cases, which appear as either $r = 0, \lambda = \phi$, or $s = 0, \lambda = \phi$, respectively.

The above colour singlet states and operators are easy to enumerate explicitly for the lowest cases $n = 1$ and $n = 2$. For the former, from the general result of section 3, the polynomial superalgebra is isomorphic to $gl(1/1)$, so the classification of states and operators can, at the same time, be viewed as a list of $gl(1/1)$ representations. The colour singlet states and operators are given in table 1.

For $n = 2$, table 2 provides a list of colour $sl(3)$ singlet $gl(2)$ representations $\{\bar{\sigma}\} \cdot \{\sigma\}$, $\{\bar{\rho}\} \cdot \{\sigma\}$, for meson, baryon, exotic baryon and dibaryon operators, and conjugates. From the partitions, it is easy to reconstruct the parameters r, s and λ used above; the fermion content (K, L) of each state or operator is given, together with the $sl(2)$ spin content (written as a reducible representation $j \times k$ corresponding to the reduction of $\{\bar{\rho}\}$ and $\{\sigma\}$ respectively), together with the dimension $N = (2j + 1)(2k + 1)$.

[23] In the context of physical applications the relevant symmetry groups are in the unitary chain, and the colour transformations belong to $SU(3)$. Here we merely count singlets of $sl(3)$.

[24] Corresponding to the reduction of the spinor representation of $so(6n)$ under branching chain $so(6n) \supset gl(3n) \supset gl(3) + gl(n)$.

Table 1. Colour singlet composite operators for $gl_2(1/\{3\} + \{\bar{3}\}) \simeq gl(1/1)$, listed by fermion content (K, L) and $gl(1)$ content $\{\bar{K}\} \cdot \{L\}$ (one-dimensonal representations with $gl(1)$ quantum number $K - L$). Mesons have $K = L$ and baryons have $K = 0$ or $L = 0$. The adjoint module has even dimension 4 and odd dimension 2 (including the two odd and two even generators of $gl(1/1)$).

(K, L)	$\{\bar{K}\} \cdot \{L\}$
(0, 0)	$\{\bar{0}\} \cdot \{0\}$
(1, 1)	$\{\bar{1}\} \cdot \{1\}$
(2, 2)	$\{\bar{2}\} \cdot \{2\}$
(3, 3)	$\{\bar{3}\} \cdot \{3\}$
(0, 3)	$\{\bar{0}\} \cdot \{3\}$
(3, 0)	$\{\bar{3}\} \cdot \{0\}$

Table 2. Colour singlet composite operators for $gl_2(2/\{3\} + \{\bar{3}\})$, listed by fermion content (K, L), $gl(2)$ content $\{\bar{\rho}\} \cdot \{\sigma\}$ for partitions in the standard $3 \times (n = 2)$ rectangular envelope and $sl(2)$ spin content $j \times k$ with dimension $N = (2j + 1)(2k + 1)$. Mesons have $K = L$, baryons and dibaryons and their conjugates have $K = 0$ or $L = 0$ and exotic baryons have $K > L > 0$ or $L > K > 0$. The adjoint module has even dimension 52 and odd dimension 40 (including the eight odd and four even generators).

(K, K)	$\{\bar{\sigma}\} \cdot \{\sigma\}$	$j \times j$	N	(K, L)	$\{\bar{\rho}\} \cdot \{\sigma\}$	$j \times k$	N
(0, 0)	$\{\bar{0}\} \cdot \{0\}$	0×0	1	(0, 3)	$\{\bar{0}\} \cdot \{3\}$	$0 \times \frac{3}{2}$	4
(1, 1)	$\{\bar{1}\} \cdot \{1\}$	$\frac{1}{2} \times \frac{1}{2}$	4	(3, 0)	$\{\bar{3}\} \cdot \{0\}$	$\frac{3}{2} \times 0$	4
(2, 2)	$\{\bar{2}\} \cdot \{2\}$	1×1	9	(1, 4)	$\{\bar{1}\} \cdot \{3, 1\}$	$\frac{1}{2} \times 1$	6
	$\{\overline{1^2}\} \cdot \{1^2\}$	0×0	1	(2, 5)	$\{\bar{2}\} \cdot \{3, 2\}$	$1 \times \frac{1}{2}$	6
(3, 3)	$\{\bar{3}\} \cdot \{3\}$	$\frac{3}{2} \times \frac{3}{2}$	16	(3, 6)	$\{\bar{3}\} \cdot \{3^2\}$	$\frac{3}{2} \times 0$	4
	$\{\overline{2, 1}\} \cdot \{2, 1\}$	$\frac{1}{2} \times \frac{1}{2}$	4	(4, 1)	$\{\overline{3, 1}\} \cdot \{1\}$	$1 \times \frac{1}{2}$	6
(4, 4)	$\{\overline{3, 1}\} \cdot \{3, 1\}$	1×1	9	(5, 2)	$\{\overline{3, 2}\} \cdot \{2\}$	$\frac{1}{2} \times 1$	6
	$\{\overline{2, 2}\} \cdot \{2, 2\}$	0×0	1	(6, 3)	$\{\overline{3^2}\} \cdot \{3\}$	$0 \times \frac{3}{2}$	4
(5, 5)	$\{\overline{3, 2}\} \cdot \{3, 2\}$	$\frac{1}{2} \times \frac{1}{2}$	4	(0, 6)	$\{\bar{0}\} \cdot \{3^2\}$	0×0	1
(6, 6)	$\{\overline{3, 3}\} \cdot \{3, 3\}$	0×0	1	(6, 0)	$\{\overline{3^2}\} \cdot \{0\}$	0×0	1

A.4 Relation between $gl_2(4/\{1^3\} + \{\overline{1^3}\})$ *and* $gl_2(4/\{1\} + \{\bar{1}\})^{\mathrm{a},\alpha,\beta}$

The approach of section 4 was to analyse abstractly the defining relations of the $gl_2(n/\{1\} + \{\bar{1}\})$ family of quadratic algebras in order to establish properties of the admissible structure constants in the absence of a particular realization. Here we reverse this philosophy and show, for the case $n = 4$, the relation between the previously considered (fermionic oscillator) $gl_2(4/\{1^3\} + \{\overline{1^3}\})$ construction and $gl_2(4/\{1\} + \{\bar{1}\})^{\mathrm{a},\alpha,\beta}$.

Consider then the generators $\bar{Q}^{ijk}$, Q_{pqr} of $gl_2(4/\{1^3\} + \{\overline{1^3}\})$ as in (14), but with the modification that the odd generators transform as tensor densities of weight w,

$$\begin{aligned} \left[E^i{}_j, \bar{Q}^{[k\ell m]}\right] &= \delta_j{}^k \bar{Q}^{[i\ell m]} + \delta_j{}^\ell \bar{Q}^{[kim]} + \delta_j{}^m \bar{Q}^{[k\ell i]} + w\delta^i{}_j \bar{Q}^{[k\ell m]} \\ \left[E^i{}_j, Q_{[pqr]}\right] &= -\delta^i{}_p Q_{[jqr]} - \delta^i{}_q Q_{[pjr]} - \delta^i{}_r Q_{[pqj]} - w\delta_i{}^j Q_{[pqr]}. \end{aligned} \tag{A.11}$$

Then, defining

$$\begin{aligned} \bar{S}^i &= \tfrac{1}{6}\epsilon^{ijk\ell} Q_{jk\ell} \qquad & Q_{jk\ell} &= -\epsilon_{jk\ell m}\bar{S}^m \\ S_i &= \tfrac{1}{6}\epsilon_{ijk\ell}\bar{Q}^{jk\ell} \qquad & \bar{Q}^{jk\ell} &= -\epsilon^{ijk\ell} S_\ell \end{aligned} \tag{A.12}$$

produces

$$\left[E^i{}_j, \bar{S}^k\right] = \delta_k{}^j \bar{S}^i - (1 + w)\delta_i{}^j \bar{S}^k \tag{A.13}$$

$$\left[E^i{}_j, S_k\right] = -\delta_i{}^k S_j + (1+w)\delta_i{}^j S_k \tag{A.14}$$

so that the choice $w = -1$ leads to standard tensor transformation rules for the rank 1 odd generators $S, \bar{S}$. Proceeding with (A.12) produces

$$\{\bar{S}^i, S_j\} = \tfrac{1}{6}[-3\{\bar{Q}^{[ik\ell]}, Q_{[jk\ell]}\} + \delta^i{}_j\{\bar{Q}^{[k\ell m]}, Q_{[k\ell m]}]$$

and use of the structure constants (14) together with the standard definitions (21) leads after use of (22) to the quoted form

$$\{\bar{S}^i, S_j\} = (J^2)^a{}_b - \tfrac{1}{2}\langle J^2\rangle\delta^a{}_b - \tfrac{1}{2}\hat{N}J^a{}_b + \left(\tfrac{3}{16}\hat{N}^2 - \tfrac{3}{4}\hat{N} + 2\right)\delta^a{}_b$$

which agrees with (21), (25) for $n = 4\left(a = -\frac{1}{2}, \mathsf{b}_1 = -\frac{1}{2}, \mathsf{c}_1 = \frac{3}{16}\right)$, together with $\mathsf{b}_2 = -1, \mathsf{c}_2 = -\frac{3}{4}$.

Finally, we comment on the role of the covariant constraints (27) in this case, and generalizations to other cases. Firstly, for the antisymmetric rank 3 case $gl_2(n/\{1^3\} + \overline{\{1^3\}})$, in the realization (14) via fermionic oscillator modes, we have directly from (12), (13) that

$$\begin{aligned} E^i{}_m\bar{Q}^{[mjk]} + E^j{}_m\bar{Q}^{[mki]} + E^k{}_m\bar{Q}^{[mij]} &= (-3\hat{N} + 3(n+1)\mathbb{1})\bar{Q}^{[ijk]} \\ Q_{[ijm]}E^m{}_k + Q_{[jkm]}E^m{}_i + Q_{[kim]}E^m{}_j &= Q_{[ijk]}(-3\hat{N} + 3(n+1)\mathbb{1}). \end{aligned} \tag{A.15}$$

Similarly in the symmetric rank 3 case $gl_2(n/\{3\} + \{\bar{3}\})$ we have from (16), (17) that

$$\begin{aligned} E^i{}_m\bar{W}^{(mjk)} + E^j{}_m\bar{W}^{(mki)} + E^k{}_m\bar{W}^{(mij)} &= (\hat{N} - 3(n-3)\mathbb{1})\bar{W}^{(ijk)} \\ W_{(ijm)}E^m{}_k + W_{(jkm)}E^m{}_i + W_{(kim)}E^m{}_j &= W_{(ijk)}(\hat{N} - 3(n-3)\mathbb{1}). \end{aligned} \tag{A.16}$$

which reflect the identity (true for any $su(3)$ tensor T_A)

$$\epsilon_{ABC}T_D - \epsilon_{BCD}T_A + \epsilon_{CDA}T_B - \epsilon_{DAB}T_C = 0.$$

Note that the left-hand sides of (A.15), (A.16) can be expressed in the form of the action of the invariants (A.9) on the appropriate odd generators (that is, the matrix action of the set of generalized Gel'fand basis generators of $gl(n)$ on $Q, \bar{Q}$, and $W, \bar{W}$), respectively. Finally for $S, \bar{S}$ as in (A.12), and $E^i{}_j$ defined as in (13), we have directly from (A.14) with $w = -1$, (A.15) that

$$(E\cdot\bar{S})^i = (4\hat{N} - 15\mathbb{1})\bar{S}^i \qquad (S\cdot E)_i = S_i(4\hat{N} - 15\mathbb{1}) \tag{A.17}$$

However, as is evident from (25), (28), in the case $\mathsf{a} = -\frac{1}{2}$, the structure coefficients are independent of the particular form of the constraint.

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